

7 Derivatives, Part I (Fundamentals)

7.1 Formal definitions

Definition: the *derivative* at a of a function f , denoted $f'(a)$, is defined as:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

There are *three intuitions* to convey about the derivative:

- *Algebraic interpretation.* The derivative tells how $f(a+h)$, the value of f as small distance from a , changes relative to $f(a)$ as h becomes very small.
- *Geometric interpretation.* Draw a line through points $(a, f(a))$ and $(a+h, f(a+h))$ for some small h . Then make h “infinitely small”. Our $f'(a)$ is the slope of that line. The tangent line is a *linear approximation* of f near a .
- *Physics interpretation.* Suppose $f(t)$ maps time to position of a car on a road (or of any object on a straight line). Suppose you want to know the average velocity between any two points in time t_1, t_2 . If $h = t_2 - t_1$ then the average velocity is $\frac{f(t_1+h) - f(t_1)}{h}$. This is the quotient of the derivative! Only with h “reduced to an infinitesimal”, producing *instantaneous velocity*.

Definition: f is called *differentiable* at a if the limit $f'(a)$ exists.

The notation $f'(a)$ suggests f' is a function. Indeed, we define f' as follows. Its domain is the set of all numbers a where f is differentiable, and its value at such a point a is the limit above. Not surprisingly, we call f' the *derivative* of f . Note that the domain of f' could be much smaller than the domain of f .

We can apply the definition of the derivative to f' yielding the *second derivative* $(f')'$, denoted f'' or $f^{(2)}$. The domain of f'' is all points a such that f' is differentiable at a . If $f''(a)$ exists, we say f is *twice differentiable* at a .

7.2 Differentiability implies continuity

We are about to prove an important theorem— that differentiability implies continuity. To do that, we begin with a convenient (simple) lemma.

Lemma: $\lim_{x \rightarrow a} f(x)$ is equivalent to $\lim_{h \rightarrow 0} f(a+h)$.

Proof. Let $x = a+h$. Then

$$\begin{aligned} & \lim_{h \rightarrow 0} f(a+h) \\ & \iff \lim_{x \rightarrow a} f(x) \\ & \iff \lim_{x \rightarrow a} f(x) \end{aligned}$$

Note that the last implication is true because

$$0 < |(x - a) - 0| < \delta \iff 0 < |x - a| < \delta$$

QED.

Theorem: if f is differentiable at a , then f is continuous at a .

Proof. We must show that:

$$\lim_{x \rightarrow a} f(x) = f(a)$$

We begin as follows:

$$\begin{aligned} \lim_{h \rightarrow 0} [f(a + h) - f(a)] &= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \cdot h \\ &= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \cdot \lim_{h \rightarrow 0} h \\ &= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \cdot 0 \\ &= 0 \end{aligned}$$

It follows that

$$\begin{aligned} \lim_{h \rightarrow 0} [f(a + h) - f(a)] &= 0 \\ \implies \lim_{h \rightarrow 0} f(a + h) - \lim_{h \rightarrow 0} f(a) &= 0 \\ \implies \lim_{h \rightarrow 0} f(a + h) &= \lim_{h \rightarrow 0} f(a) = f(a) \end{aligned}$$

7.3 Low-level proofs

In the next chapter we prove theorems that make finding derivatives for many classes of functions easy. But for now we show four low-level derivations directly from the definition. Here we will be looking at constant functions, linear functions, quadratic, and cubic functions.

Constant functions

Let $f(x) = c$. Then:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = 0$$

Thus f is differentiable at a for every number a , and $f'(a) = 0$.

Linear functions

Let $f(x) = cx + d$. Then:

$$\begin{aligned}f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\&= \lim_{h \rightarrow 0} \frac{c(a+h) + d - (ca + d)}{h} \\&= \lim_{h \rightarrow 0} \frac{ch}{h} = c\end{aligned}$$

Thus f is differentiable at a for every number a , and $f'(a) = c$.

Quadratic functions

Let $f(x) = x^2$. Then:

$$\begin{aligned}f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\&= \lim_{h \rightarrow 0} \frac{(a+h)^2 - a^2}{h} \\&= \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 - a^2}{h} \\&= \lim_{h \rightarrow 0} \frac{2ah + h^2}{h} \\&= \lim_{h \rightarrow 0} 2a + h \\&= \lim_{h \rightarrow 0} 2a\end{aligned}$$

Thus f is differentiable at a for every number a , and $f'(a) = 2a$.

Cubic functions

Let $f(x) = x^3$. Then:

$$\begin{aligned}f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\&= \lim_{h \rightarrow 0} \frac{(a+h)^3 - a^3}{h} \\&= \lim_{h \rightarrow 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h} \\&= \lim_{h \rightarrow 0} \frac{3a^2h + 3ah^2 + h^3}{h} \\&= \lim_{h \rightarrow 0} 3a^2 + 3ah + h^2 \\&= 3a^2\end{aligned}$$

Thus f is differentiable at a for every number a , and $f'(a) = 3a^2$.

7.4 Non-differentiability

Continuous functions are “nice”. Functions that are differentiable everywhere are “nicer”. Functions that are differentiable everywhere and whose first derivative is differentiable everywhere are nicer still. Thus to fully understand the derivative we must understand examples where it does not exist.

We now turn our attention to functions that aren’t differentiable at some points a . We first look at four simple examples where there isn’t everywhere a first derivative. We then turn our attention to a more subtle example— a function that’s differentiable in the first, but not everywhere in the second derivative.

First derivative

Example 1

Let $f(x) = |x|$. Consider $f'(0)$:

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h}$$

Observe that $\lim_{h \rightarrow 0^+} \frac{|h|}{h} = 1$ and $\lim_{h \rightarrow 0^-} \frac{|h|}{h} = -1$. This $\lim_{h \rightarrow 0} \frac{|h|}{h}$ does not exist, and f is not differentiable at 0. Note that f is differentiable at every other point: $f'(a) = -1$ for $a < 0$ and $f'(a) = 1$ for $a > 0$.

Example 2

Let f be defined as follows:

$$f(x) = \begin{cases} x^2, & x \leq 0 \\ x, & x \geq 0 \end{cases}$$

Now consider $f'(0)$:

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h)}{h}$$

Observe that

$$\frac{f(h)}{h} = \begin{cases} \frac{h^2}{h} = h, & h \leq 0 \\ \frac{h}{h} = 1, & h \geq 0 \end{cases}$$

Therefore $\lim_{h \rightarrow 0^-} \frac{f(h)}{h} = 0$ and $\lim_{h \rightarrow 0^+} \frac{f(h)}{h} = 1$. Thus $\lim_{h \rightarrow 0} \frac{f(h)}{h}$ does not exist, and f is not differentiable at 0.

Example 3

Let $f(x) = \sqrt{|x|}$. Consider $f'(0)$:

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{|h|}}{h}$$

Observe that

$$\frac{\sqrt{|h|}}{h} = \begin{cases} \frac{\sqrt{-h}}{h} = -\frac{1}{\sqrt{-h}}, & h < 0 \\ \frac{\sqrt{h}}{h} = \frac{1}{\sqrt{h}}, & h > 0 \end{cases}$$

Therefore $\lim_{h \rightarrow 0^+} \frac{\sqrt{|h|}}{h} = \infty$ and $\lim_{h \rightarrow 0^-} \frac{\sqrt{|h|}}{h} = -\infty$. Thus $\lim_{h \rightarrow 0} \frac{\sqrt{|h|}}{h}$ does not exist, and f is not differentiable at 0.

Example 4

Let $f(x) = \sqrt[3]{x}$. Here $\frac{\sqrt[3]{h}}{h}$ plays out as follows:

$$\frac{\sqrt[3]{h}}{h} = \frac{h^{1/3}}{h} = \frac{1}{h^{2/3}} = \frac{1}{(\sqrt[3]{h})^2}$$

This expression becomes arbitrarily large as h goes to 0, i.e. $\lim_{h \rightarrow 0} \frac{1}{(\sqrt[3]{h})^2} = \infty$. Thus f is not differentiable at zero (or put differently, the tangent line to f at 0 is vertical).

Second derivative

We now come to our more subtle example— a function that’s differentiable in the first but not everywhere in the second derivative:

Example 1

Let

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x \leq 0 \end{cases}$$

As we’ve seen in the quadratic functions example above, $\frac{dx^2}{dx} = 2x$. By very similar logic, $\frac{d(-x^2)}{dx} = -2x$. Thus $f'(a) = 2a$ if $a \geq 0$, and $f'(a) = -2a$ if $a \leq 0$. Or, put differently, $f'(x) = 2|x|$.

So, $f'(0) = 0$. But what about $f''(0)$? We’ve already seen that $g(x) = |x|$ is not differentiable, and by very similar logical, $f''(0)$ does not exist! So even a “smooth looking” function may not have a second derivative— a fact that implies existence of a second derivative is a strong criterion for a function to satisfy.

7.5 Tangent lines

Spivak now handles a question— how many times does a tangent line to f at a intersect the graph of f ? He doesn’t yet motivate the question, though I suspect he will at a later time (at which point I will come back and make a note of it here). To answer this question we must first find the linear equation that describes the tangent line, which we will now do. We’ll then address the question of intersection for quadratic and cubic functions.

Point-slope form

The slope m of a line is determined by

$$m = \frac{y - y_1}{x - x_1}$$

For a function f differentiable at a , the slope of a tangent line at a is $m = f'(a)$. We also know one of the points on the line— $(a, f(a))$ (the point where the tangent line intersects with f). Plugging that in we get

$$\begin{aligned} f'(a) &= \frac{y - f(a)}{x - a} \\ \implies y - f(a) &= f'(a)(x - a) \\ \implies y &= f'(a)(x - a) + f(a) \end{aligned}$$

This gives us a linear equation for the tangent line— a *linear approximation* of f near a .

Intersections

Quadratic functions. For $f(x) = x^2$ we've seen that $f'(a) = 2a$. Plugging that into $y = f'(a)(x - a) + f(a)$ we get:

$$\begin{aligned} y &= 2a(x - a) + a^2 \\ &= 2ax - a^2 \end{aligned}$$

Let $g(x) = 2ax - a^2$. We can now solve for all x such that $f(x) = g(x)$:

$$\begin{aligned} f(x) &= g(x) \\ \implies x^2 &= 2ax - a^2 \\ \implies x^2 - 2ax + a^2 &= 0 \\ \implies (x - a)^2 &= 0 \\ \implies x &= a \end{aligned}$$

So the only solution is $x = a$, therefore the only point of intersection is $(a, f(a) = g(a) = a^2)$.

Cubic functions. For $f(x) = x^3$ we've seen that $f'(a) = 3a^2$. Plugging that into $y = f'(a)(x - a) + f(a)$ we get:

$$\begin{aligned} y &= 3a^2(x - a) + a^3 \\ &= 3a^2x - 2a^3 \end{aligned}$$

Let $g(x) = 3a^2x - 2a^3$. We can now solve for all x such that $f(x) = g(x)$:

$$\begin{aligned} f(x) &= g(x) \\ x^3 &= 3a^2x - 2a^3 \\ x^3 - 3a^2x + 2a^3 &= 0 \end{aligned}$$

Factoring this in Mathematica, we get

$$(a - x)^2(2a + x) = 0$$

Thus one of the solutions is $x = a$ (we already knew (a, a^3) is a point of intersection– we explicitly constructed the tangent line at this point). The other solution is $x = -2a$, and thus the other point of intersection is $(-2a, -8a^3)$.